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Note on restricted Gröbner fans

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ABSTRACT

In this paper we look at the Gröbner fan of a nonzero proper ideal in a polynomial ring. We give a concrete description of the restricted Gröbner fan of the ideal with respect to a tuple of terms and show how to compute it. Moreover, under a certain condition, we provide the connection between the restricted Gröbner fan of an ideal and the Gröbner fan of the corresponding elimination module.

1. INTRODUCTION

Let K be a field and let I be a nonzero proper ideal in a polynomial ring $P = K[x_1,...,x_n]$ over K. The notion of a Gröbner fan of I was introduced and studied by Mora and Robbiano (1988). It is a finite set of all distinct marked reduced Gröbner bases of I which play a fundamental role in Computer Algebra. The theory of Gröbner fans has been extensively developed and generalized in various contexts (see, for instance, (Assi et al., 2000; Bahloul & Takayama, 2007; Kreuzer et al., 2022; Sturmfels, 1996 and the references therein).

Based on the discovery of separating re-embeddings of non-trivial examples, in (Kreuzer et al., 2023), the authors developed the theory of Gröbner fans further by bringing the notion of the restricted T-Gröbner fan of I for a tuple of terms T in P into this topic. As a subset of the Gröbner fan of I, the restricted T-Gröbner fan of I consists of all marked reduced Gröbner bases of I whose leading terms contain T . When T is a tuple of indeterminates, there is a useful 1-1 correspondence between the restricted T-Gröbner fan of I and the Gröbner fan of its elimination ideal. This allows us to compute the restricted T-Gröbner fan of I without the need for a full Gröbner fan computation and this helps to find separating re-embeddings of I efficiently. In general, for a tuple of terms T in P , there is typically no connection between the restricted T-Gröbner fan of I and the Gröbner fan of its elimination ideal. In this situation, one can define a free module $\mathcal{F}$ over the polynomial ring $\hat{P}$ in indeterminates, which do not divide any term of T , and then looks at the elimination module $I \cap \mathcal{F}$ instead of the elimination ideal $I \cap \hat{P}$ (Kreuzer et al., 2023; Section 7). Then there exists a connection map between the restricted T-Gröbner fan of I and the Gröbner fan of $I \cap \mathcal{F}$. Naturally, we may ask whether the connection map is injective or surjective. In this article, we explicitly describe the restricted T-Gröbner fan and discuss about the above question.

The article is organized as follows. Section 2 reviews the necessary theory of reduced Gröbner bases, Gröbner fans, and the 1-1 correspondence between the restricted T-Gröbner fan of I and the Gröbner fan of its elimination ideal when T consists of indeterminates (see Theorem 2.7). In Section 3, we first look at the restricted T-Gröbner fan of I for a tuple of terms in P and indicate how to compute it (see Proposition 3.3). Then we describe the T-separating module re-embedding of I. In the last section, we first give equivalent conditions for the finiteness of the module $\mathcal{F}$ (see Proposition 4.2) and explicitly describe the reduced Gröbner bases of the elimination module $I \cap \mathcal{F}$ (see Proposition 4.3). Under a certain condition, we show that the connection map from the restricted T-Gröbner fan of I to the Gröbner fan of the elimination module $I \cap \mathcal{F}$ is an injection (see Proposition 4.6). Finally, we derive consequences for the tuple T of pure powers of

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indeterminates. The notation and definitions in this article follow (Kreuzer & Robbiano, 2000; 2005). The calculations underlying most examples were performed using the computer algebra system ApCoCoA (n.d.).

2. PRELIMINARIES

Let $P = K[x_1,...,x_n]$ be a polynomial ring over a field K, let $\mathbb{T}^n = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid (\alpha_1,...,\alpha_n) \in \mathbb{N}^n\}$ be the set of all terms in P, let I be a nonzero proper ideal of P, and let σ be a term ordering on P. The support of a nonzero polynomial $f \in P$ is the set of its terms and is denoted by $\mathrm{Supp}(f)$, and the leading term and leading coefficient of $f \in P\setminus\{0\}$ with respect to σ are denoted by $\mathrm{LT}_\sigma(f)$ and $\mathrm{LC}_\sigma(f)$, respectively. The leading term ideal of I is the monomial ideal generated by all leading terms of its nonzero polynomials, denoted by $\mathrm{LT}_\sigma(I)$.

Recall that a finite set of nonzero polynomials $G = \{g_1,...,g_r\}$ in I is a *σ-Gröbner basis* of I if $\mathrm{LT}_\sigma(I) = \langle \mathrm{LT}_\sigma(g_1),...,\mathrm{LT}_\sigma(g_r)\rangle$. If G is a σ-Gröbner basis of I then it is also a system of generators of I; and every ideal in P has a σ-Gröbner basis. These notions and properties are similarly valid for a submodule $M \subseteq P^s$ with positive integer $s \geq 1$. For a module term ordering σ on the set of terms $\mathbb{T}^n\langle e_1,...,e_s\rangle$, there is a unique one among all σ-Gröbner bases of M, which is defined as follows (Kreuzer & Robbiano, 2000, Definition 2.4.12).

Definition 2.1. Let σ be a module term ordering of $\mathbb{T}^n\langle e_1,...,e_s\rangle$, let $G = \{g_1,...,g_r\} \subseteq P^s\setminus\{0\}$ and $M = \langle g_1,...,g_r\rangle$. We say that G is a **reduced σ-Gröbner basis** of M if the following conditions are satisfied.

(a) For $i = 1,...,r$, $\mathrm{LT}_\sigma(g_i) = 1$.

(b) The set $\{\mathrm{LT}_\sigma(g_1),...,\mathrm{LT}_\sigma(g_s)\}$ is the minimal system of generators of $\mathrm{LT}_\sigma(M)$.

(c) For $i = 1,...,r$, $\mathrm{Supp}(g_i - \mathrm{LT}_\sigma(g_i)) \cap \mathrm{LT}_\sigma(M) = \emptyset$.

The existence and uniqueness of a reduced σ-Gröbner basis of M is guaranteed by (Kreuzer & Robbiano, 2000, Theorem 2.4.13). Clearly, the reduced Gröbner basis of M depends on the module term ordering σ. This leads us to the notion of marked reduced Gröbner bases which was introduced in (Mora & Robbiano, 1988).

Definition 2.2. Let σ be a module term ordering of $\mathbb{T}^n\langle e_1,...,e_s\rangle$, and let M be a proper submodule of P^s.

(a) The marked reduced σ-Gröbner basis of M is a set of pairs

$$\overline{G} = \{(\mathrm{LT}_\sigma(g_1), g_1),...,(\mathrm{LT}_\sigma(g_r), g_r)\},$$

where $G = \{g_1,...,g_r\}$ is the reduced σ-Gröbner basis of M.

(b) The **Gröbner fan** of M is the set of all distinct marked reduced Gröbner bases of M. It is denoted by $\mathrm{GFan}(M)$.

When I is a proper ideal of P, it is well-known that $\mathrm{GFan}(I)$ is finite and algorithmically computed (Mora & Robbiano, 1988).

Example 2.3. Let $\mathbf{P} = \mathbb{Q}[x,y,z]$, and let $I = \langle f_1, f_2\rangle$, where $f_1 = x^2 + yz + z^2$ and $f_2 = y^2z + yz^2 - z + 1$. Then the Gröbner fan of I consists of 6 marked reduced Gröbner bases corresponding to 6 different leading term ideals of I which depend on the term orderings on $\mathbb{T}^3$. They are given by

$$\begin{aligned}
&\{(x^2, x^2+yz+z^2), (y^2z, y^2z+yz^2-z+1)\},\\
&\{(x^2, x^2+z^2+yz), (yz^2, yz^2+y^2z-z+1)\},\\
&\{(yz, yz+x^2+z^2), (x^2y, x^2y+z-1), (x^4, x^4+x^2z^2-z^2+z)\},\\
&\{(yz, yz+z^2+x^2), (x^2y, x^2y+z-1), (x^2z^2, x^2z^2+x^4-z^2+z)\},\\
&\{(z^2, z^2+yz+x^2), (x^2y, x^2y+z-1)\},\\
&\{(z, z+x^2y-1), (x^4y^2, x^4y^2-x^2y^2-2x^2y+y+x^2+1)\}.
\end{aligned}$$

In (Kreuzer et al., 2022), to investigate separating re-embeddings of the ideal I in P, the authors looked at a subset of $\mathrm{GFan}(I)$ with respect to a tuple of indeterminates which is given as follows.

Definition 2.4. If $Z = (z_1,...,z_s)$ is a tuple of distinct indeterminates such that $z_i \in \{x_1,...,x_n\}$ for $i = 1,...,s$, the set $\mathrm{GFan}_Z(I)$ of all marked reduced σ-Gröbner bases

$$\overline{G} = \{(\mathrm{LT}_\sigma(g_1),g_1),...,(\mathrm{LT}_\sigma(g_r),g_r)\}$$

of I such that for $j = 1,...,s$ there are indices $i_j \in \{1,...,r\}$ with $z_j = \mathrm{LT}_\sigma(g_{i_j})$ is called the ***Z*-restricted Gröbner fan** of I.

Remark 2.5. Let $Z = (z_1,...,z_s)$ be a tuple of distinct indeterminates in the set $\{x_1,...,x_n\}$, let Y be the complement of Z in $\{x_1,...,x_n\}$, and let $\hat{P} = K[Y]$. A Z-restricted marked reduced σ-Gröbner basis $\overline{G}$ in $\mathrm{GFan}_Z(I)$ can be written in the form

$$\overline{G} = \{(z_1, z_1 - h_1),\ldots,(z_s, z_s - h_s), (\mathrm{LT}_\sigma(g_{s+1}), g_{s+1}),\ldots, (\mathrm{LT}_\sigma(g_r), g_r)\}$$

where $h_1,\ldots,h_s, g_{s+1},\ldots, g_r \in \widehat{P}$. This basis induces a K-algebra isomorphism $\Phi : P / I \to \widehat{P} / (I \cap \widehat{P})$ given by $\Phi(\bar{x}_i) = \bar{x}_i$ if $x_i \notin Z$ and $\Phi(\bar{x}_i) = \bar{h}_j$ if $x_i = z_j \in Z$. This re-embedding is known as a Z-separating re-embedding from P / I to $\widehat{P} / (I \cap \widehat{P})$ in the sense of (Kreuzer et al., 2022; Definition 2.5).

Example 2.6. In $P = \mathbb{Q}[x,y,z]$, consider the ideal $I = \langle f_1, f_2 \rangle$ with $f_1 = x^2 + yz + z^2$ and $f_2 = y^2z + yz^2 - z +1$ as in Example 2.3. For $Z_1 = (x)$ we have $\mathrm{GFan}_{Z_1}(I) = \emptyset$. For $Z_2 = (z)$, $\mathrm{GFan}_{Z_2}(I)$ contains only one element given by

$$\overline{G} = \{(z,z + x^2y - 1), (x^4y^2, x^4y^2 - x^2y^2 - 2x^2y + y + x^2 + 1)\}.$$

This basis corresponds to an elimination for Z_2, and consequently the elimination ideal is $I \cap \widehat{P} = \langle x^4y^2 - x^2y^2 - 2x^2y + y + x^2 + 1\rangle$. In this case we have the Z_2-separating re-embedding Φ: $P / I \to \widehat{P} /(I \cap \widehat{P})$ given by $\Phi(\bar{x}) = \bar{x}$, $\Phi(\bar{y}) = \bar{y}$ and $\Phi(\bar{z}) = -\bar{x}^2\bar{y} + 1$.

An interesting result on this notion is the following connection between the Z-restricted Gröbner fan and the Gröbner fan of the elimination ideal $I \cap \widehat{P}$ of I (Kreuzer et al., 2023, Theorem 5.5).

Theorem 2.7. *Let $Z = (z_1,\ldots,z_s)$ be a tuple of distinct indeterminates in the set $\{x_1,\ldots,x_n\}$, let Y be the complement of Z in $\{x_1,\ldots,x_n\}$, and let $\widehat{P} = K[Y]$. Let $I \subsetneq \mathfrak{M} = \langle x_1,\ldots,x_n\rangle$ be an ideal of P such that $\mathrm{GFan}_Z(I) \neq \emptyset$. Then the map $\Gamma_Z : \mathrm{GFan}_Z(I) \to \mathrm{GFan}(I \cap \widehat{P})$ sending*

$$\overline{G} =\{(z_1,g_1),\ldots,(z_s,g_s), (\mathrm{LT}_\sigma(g_{s+1}), g_{s+1}),\ldots,(\mathrm{LT}_\sigma(g_r), g_r)\} \in \mathrm{GFan}_Z(I)$$

to $\{(\mathrm{LT}_\sigma(g_{s+1}), g_{s+1}),\ldots,(\mathrm{LT}_\sigma(g_r), g_r)\} \in \mathrm{GFan}(I \cap \widehat{P})$ is a bijection.

This result enables the efficient computation of the Z-restricted Gröbner fan $\mathrm{GFan}_Z(I)$ without the need for a full Gröbner fan computation (Kreuzer et al., 2023, Example 5.7).

3. GENERALIZED RESTRICTED GR¨ OBNER FANS

As before, let I be a nonzero proper ideal in $P = K[x_1,\ldots,x_n]$, and let $T = (t_1,\ldots,t_s)$ be a tuple of distinct terms in $\mathbb{T}^n$. It is also possible to require that some leading terms of a reduced Gröbner basis are predescribed terms given by T rather than predescribed indeterminates given by a tuple of distinct indeterminates. This more general type of restriction leads to the following concept introduced in (Kreuzer et al., 2023).

Definition 3.1. The set of all marked reduced Gröbner bases of I of the form

$$\overline{G} = \{(\mathrm{LT}_\sigma(g_1), g_1), \ldots, (\mathrm{LT}_\sigma(g_r), g_r)\} \tag{1}$$

such that for $i = 1,\ldots,s$ there exist $j_i \in \{1,\ldots,r\}$ with $\mathrm{LT}_\sigma(g_{j_i}) = t_i$ is called the ***T*-restricted Gröbner fan** of I and is denoted by $\mathrm{GFan}_T(I)$.

Remark 3.2. Note that if $t_i \mid t_j$ for some $i,j \in \{1,\ldots,s\}$ with $i \neq j$ then $\mathrm{GFan}_T(I) = \emptyset$, since a marked reduced Gröbner basis $\overline{G}$ of I as above always has $\mathrm{LT}_\sigma(g_i) \nmid \mathrm{LT}_\sigma(g_j)$ for all $i \neq j$.

To compute $\mathrm{GFan}_T(I)$, one can apply the following algorithm.

Proposition 3.3 (**Computing** $\mathrm{GFan}_T(I)$). *Let I be a nonzero proper ideal in P, and let $T = (t_1, \ldots, t_s)$ be a tuple of distinct terms in $\mathbb{T}^n$ with $t_i \nmid t_j$ for all $i \neq j$. The T-restricted Gröbner fan of I can be computed by the following instructions.*

- *(a) Let Z' be the set of indeterminates in T , set $T' = T \setminus Z'$, and $\Sigma = \emptyset$.*
- *(b) Compute the set $\mathrm{GFan}_{Z'}(I)$ using Theorem 2.7.*
- *(c) For each $\overline{G} \in \mathrm{GFan}_{Z'}(I)$, check if $T' \subseteq \mathrm{LT}_\sigma(\overline{G}) = \{\mathrm{LT}_\sigma(g) \mid (\mathrm{LT}_\sigma(g),g) \in \overline{G}\}$. If this is true, append $\overline{G}$ to Σ.*
- *(d) Return the set Σ.*

Proof. We know that $\overline{G} \in \mathrm{GFan}_{Z'}(I)$ if and only if $Z' \subseteq \mathrm{LT}_\sigma(\overline{G})$. Also, we have $\mathrm{GFan}_T(I) \subseteq \mathrm{GFan}_{Z'}(I)$, as $Z' \subseteq T$. Step (2) shows how to compute $\mathrm{GFan}_{Z'}(I)$, and Step (3) ensures that $\Sigma \subseteq \mathrm{GFan}_T(I)$. If $\overline{G} \in \mathrm{GFan}_T(I)$ then $T \subseteq \mathrm{LT}_\sigma(\overline{G})$, and hence $\overline{G} \in \mathrm{GFan}_{Z'}(I)$ and $T' \subseteq \mathrm{LT}_\sigma(\overline{G})$, that is $\overline{G} \in \Sigma$. This completes the proof.

Example 3.4. Let us consider the ideal I in $P = \mathbb{Q}[x,y,z]$ generated by $f_1 = x^2 + yz + z^2$ and $f_2 = y^2 - x^2y - zx$. Then the Gröbner fan $\mathrm{GFan}(I)$ consists of 18 marked reduced Gröbner bases.

- If $T =\{x\}$ then $\mathrm{GFan}_T(I) = \emptyset$.
- For $T =\{x^2\}$, $\mathrm{GFan}_T(I)$ has 6 T-restricted marked reduced Gröbner bases. One of these bases is

$\overline{G} = \{(x^2, g_1), (xy, g_2), (xy^2, g_3), (y^4z^2, g_4)\}$, where

$$g_1 = x^2 + yz + z^2,$$
$$g_2 = xz - y^2z - yz^2 - y^2,$$
$$g_3 = xy^2 + y^4z + 2y^3z^2 + y^2z^3 + y^4 + y^3z + yz^2 + z^3,$$
$$g_4 = y^4z^2 + 2y^3z^3 + y^2z^4 + 2y^4z + 2y^3z^2 + y^4 + yz^3 + z^4.$$

Moreover, we also have $I \cap \mathbb{Q}[y,z] = \langle g_4\rangle$. Clearly, $g_1 = f_1$ and $g_1 - x^2$ has no terms divisible by x^2, but we can not define an embedding from P/I to $\mathbb{Q}[y,z]/\langle g_4\rangle$. Instead we may define $\mathcal{F} = \mathbb{Q}[y,z] \oplus x\mathbb{Q}[y,z]$ as a free $\mathbb{Q}[y,z]$-module of rank 2, and from the intersection $I \cap \mathcal{F}$. Then we can consider the map

$$\Phi: P/I \to \mathcal{F}/(I \cap \mathcal{F}),\ f + I \to \mathrm{NF}_{\langle f_1\rangle}(f) + I \cap \mathcal{F}.$$

In this case one can check that the map Φ is an isomorphism of $\mathbb{Q}[y,z]$-modules.

In what follows, let $T = (t_1,...,t_s)$ be a tuple of distinct terms in $\mathbb{T}^n$ such that $\mathrm{GFan}_T(I) \neq \emptyset$, let Z be the set of indeterminates in $\{x_1,...,x_n\}$ dividing one of the terms on T, and let Y be the complement of Z in $\{x_1,...,x_n\}$. Furthermore, let $\mathbb{T}(Z)$ and $\mathbb{T}(Y)$ be the sets of all terms in $K[Z]$ and in $K[Y]$, respectively, and put $\mathcal{O}_T := \mathbb{T}(Z) \setminus \langle t_1,...,t_s\rangle$ and $\hat{P} := K[Y]$. Let

(2) $$\bar{G} = \{(t_1, t_1 - h_1), ..., (t_s, t_s - h_s), (\mathrm{LT}_\sigma(g_{s+1}), g_{s+1}),..., (\mathrm{LT}_\sigma(g_r), g_r)\}$$

be an element of $\mathrm{GFan}_T(I)$, where $h_1,...,h_s, g_{s+1},..., g_r \in P$. We form $H = (t_1-h_1,...,t_s-h_s)$ and the free $\hat{P}$-module $\mathcal{F} = \oplus_{t\in\mathcal{O}_T} t\cdot \hat{P}$.

Then the normal form map

$$\mathrm{NF}_H : P \to \mathcal{F},\ f \to \mathrm{NF}_H(f)$$

is a homomorphism of $\hat{P}$-modules such that $\mathrm{NF}_H(I) \subseteq I \cap \mathcal{F}$. Indeed, for $f \in P$, the support of $\mathrm{NF}_H(f)$ is contained in $\mathcal{O}_T \cdot \mathbb{T}(Y)$, and so $\mathrm{NF}_H(f) \in \mathcal{F}$, in particular, $\mathrm{NF}_H(f) \subseteq I \cap \mathcal{F}$ whenever $f \in I$. Moreover, NF_H is $\hat{P}$-linear, since every indeterminate in Y does not divide any term in T. This map induces a homomorphism of $\hat{P}$-modules

$$\Phi_T : P/I \to \mathcal{F}/(I \cap \mathcal{F}).$$

It is called the T-**separating module re-embedding** of the ideal I.

Proposition 3.5 (Kreuzer et al., 2023, Proposition 7.3). *The map Φ_T is an isomorphism of $\hat{P}$-modules.*

It is interesting to know when $\mathcal{F}$ is of finite rank and what a reduced Gröbner basis of $I \cap \mathcal{F}$ is. Let σ' be the restriction of σ to $\mathcal{F}$. Then σ' is a $\hat{P}$-module term ordering on $\mathcal{F}$. The following result follows from (Kreuzer et al., 2022; Proposition 7.3). For convenience for the reader, we include its proof here.

Proposition 3.6. *Let $\bar{G}$ be an element of $GFan_T(I)$ given by (2). Then the set*

$$\bar{G} = \{ug_i \in \mathcal{F} \mid u \in \mathbb{T}(Z), i \in \{s+1,...,r\}\}$$

is a σ'-Gröbner basis of the $\hat{P}$-submodule $I \cap \mathcal{F}$ of $\mathcal{F}$.

Proof. By there ducedness of $\bar{G}$, the polynomials $g_{s+1},...,g_r$ are contained in the module $I \cap \mathcal{F}$. For $f \in I \cap \mathcal{F}$ and $G = \{g_1,...,g_r\}$, we have $\mathrm{NF}_G(f) = 0$. Since the support of f is contained in $\mathcal{F}$, only $g_{s+1},...,g_r$ can be involved in the reduction steps $f \xrightarrow{G} 0$. Suppose that in each reduction step a polynomial of the form $ug_i \in \mathcal{F}$ with $u \in \mathbb{T}^n$ and $i \in \{s+1,...,r\}$ is subtracted. Wewrite $u = u'u''$ with $u' \in \mathbb{T}(Z)$ and $u'' \in \mathbb{T}(Y)$. Then the reduction step subtracts a $\hat{P}$-multiple of $u'g_i \in \mathcal{F}$. Hence f can be reduced to zero in the $\hat{P}$-module $I \cap \mathcal{F}$ via $\tilde{G}$.

4. FINITENESS AND SPECIAL RESTRICTED GRÖBNER FANS

As in the previous section, we let I be a nonzero proper ideal in P and let $T = (t_1,...,t_s)$ be a tuple of distinct terms in $\mathbb{T}^n$ such that $\mathrm{GFan}_T(I) \neq \emptyset$. Notice that the $\hat{P}$-module $\mathcal{F}$ is not necessarily finitely generated.

Example 4.1. In $P = \mathbb{Q}[x,y,z]$, let $T = (xy)$, and consider the ideal I generated by $f_1 = x^2 + yz + z^2$ and $f_2 = xy - y^2 + x$. Then $\bar{G} = \{(z^2, f_1), (xy, f_2)\}$ is a marked reduced Gröbner basis of I with respect to the term ordering $\sigma = \mathrm{Lex}(z > x > y)$. Here we write $f_1 = z^2 + yz + x^2$ and $f_2 = xy + x - y^2$. Then $Z = \{x,y\}$ and $Y = \{z\}$ and $\hat{P} = \mathbb{Q}[z]$.

- We have $\mathcal{O}_T = \{1, x, y, x^2, y^2, x^3, y^3, ...\}$.
- The module $\mathcal{F}$ is the free $\hat{P}$-module

$$\mathcal{F} = \hat{P} \oplus x\hat{P} \oplus y\hat{P} \oplus x^2\hat{P} \oplus y^2\hat{P} \oplus \cdots$$

By Proposition 3.6, the set $G' = \{f_1\}$ is a σ'-Gröbner basis of the $\hat{P}$-module $I \cap \mathcal{F}$, and hence $I \cap \mathcal{F} = \hat{P} \cdot f_1$.

- The T-separating module re-embedding of I is given by the $\hat{P}$-isomorphism $P / I \cong \mathcal{F} / \hat{P} \cdot f_1$.

The $\hat{P}$-module $\mathcal{F}$ is finitely generated in the following case.

Proposition 4.2. *In the above setting, the following conditions are equivalent:*

(a) $\mathcal{O}_T$ is finite.

(b) $\mathcal{F}$ is a free $\hat{P}$-module of finite rank.

(c) $\sqrt{\langle t_1, \dots, t_s \rangle} = \langle Z \rangle$.

Proof. The equivalence of (a) and (b) follows from the definition of $\mathcal{F}$. Now we prove the equivalence of (a) and (c). In $K[Z]$, $\mathcal{O}_T$ is the quotient basis of the monomial ideal $\langle t_1,\dots,t_s \rangle$. By (Kreuzer & Robbiano, 2000, Theorem 3.7.1), $\mathcal{O}_T$ is finite if and only if $z_i^{\alpha i} \in \langle t_1,\dots,t_s \rangle$ for $z_i \in Z$ and $\alpha_i \geq 1$ and for all $i = 1,\dots,\#Z$. The last condition is equivalent to the fact that $\sqrt{\langle t_1, \dots, t_s \rangle} = \langle Z \rangle$. □

Proposition 4.3. *Let $\bar{G} \in GFan_T(I)$ be an element given in (2), let $G'' = \{g_{s+1},\dots,g_r\}$. Suppose that $\mathcal{O}_T$ is finite.*

(a) Reducing the set

$$G' = \{ug_i \in \mathcal{F} \mid u \in \mathcal{O}_T,\ i \in \{s + 1,\dots,r\}\}$$

gives the σ'-Gröbner basis G^* of the $\hat{P}$-submodule $I \cap \mathcal{F}$ of $\mathcal{F}$.

(b) If $G'' \subseteq \hat{P}$ then G'' is the reduced σ-Gröbner basis of $I \cap \hat{P}$ and $G' = G^*$.

Proof. (a) By Proposition 3.6, we know that

$$\tilde{G} = \{ug_i \in \mathcal{F} \mid u \in \mathbb{T}(Z),\ I \in \{s+1,\dots,r\}\}$$

is a σ'-Gröbner basis of the $\hat{P}$-module $I \cap \mathcal{F}$. Also, $ug_i \notin \mathcal{F}$ for any $u \in \mathbb{T}(Z) \setminus \mathcal{O}_T$. So, G' is also a σ'-Gröbner basis of $I \cap \mathcal{F}$. Since $\mathcal{O}_T$ is finite, G' is also finite. By (Kreuzer & Robbiano, 2000, Theorem 2.4.13), we can reduce the basis G' to obtain the σ'-Gröbner basis G^* of the submodule $I \cap \mathcal{F}$.

(b) Suppose that $G'' \subseteq \hat{P}$. First we will indicate that G'' is the reduced σ-Gröbner basis of $I \cap \hat{P}$. Clearly, we have $G'' \subseteq I \cap \hat{P}$. Let $f \in I \cap \hat{P}$. Then $\mathrm{NF}_G(f) = 0$. No term of f is divisible by ti for $i = 1,\dots,s$, hence f can be reduced to zero in $\hat{P}$ using G''. This means that $\mathrm{NF}_{G''}(f) = 0$, and hence G'' is a σ-Gröbner basis of $I \cap \hat{P}$. Since G'' is fully reduced, it is the reduced σ-Gröbner basis of $I \cap \hat{P}$.

When $G'' \subseteq \hat{P}$, we have

$$G' = \{ug_i \mid u \in \mathcal{O}_T,\ i \in \{s+1,\dots,r\}\}$$

Now we show that G' is in fact reduced. Because $\mathcal{O}_T$ is a $\hat{P}$-basis of $\mathcal{F}$, we see that $\mathrm{LC}_{\sigma'}(ug_i) = 1$ for every $ug_i \in G'$. Moreover, the set $\mathrm{LT}_{\sigma'}(G')$ is a system of generators of $\mathrm{LT}_{\sigma'}(I \cap \mathcal{F})$. For fixed terms $u,u' \in \mathcal{O}_T$ and $i,j \in \{s + 1,\dots,r\}$ with $i \neq j$, we have $\mathrm{LT}_{\sigma'}(ug_i) = u\mathrm{LT}_{\sigma}(g_i)$ and $\mathrm{LT}_{\sigma'}(u'g_j) = u'\mathrm{LT}_{\sigma}(g_i)$. Since $\mathrm{LT}_{\sigma}(g_i) \in \hat{P}$ for all $i \in \{s + 1,\dots,r\}$, we get $\mathrm{LT}_{\sigma'}(ug_i) \nmid \mathrm{LT}_{\sigma'}(u'g_j)$ and $\mathrm{LT}_{\sigma'}(u'g_j) \nmid \mathrm{LT}_{\sigma'}(ug_i)$. Subsequently, $\mathrm{LT}_{\sigma'}(G')$ is a minimal system of generators of $\mathrm{LT}_{\sigma'}(I \cap \mathcal{F})$. Furthermore, no term in the support of $ug_i - \mathrm{LT}_{\sigma'}(ug_i) = u(g_i - \mathrm{LT}_{\sigma'}(g_i))$ is divisible by $\mathrm{LT}_{\sigma'}(u'g_j)$. Thus, no term in the support of $ug_j - \mathrm{LT}_{\sigma'}(ug_j)$ is contained in $\mathrm{LT}_{\sigma'}(G')$, and this yields that $G' = G^*$ is the reduced σ′-Gröbner basis of $I \cap \mathcal{F}$. □

Remark 4.4. If $T = Z$ is a tuple of indeterminates then $G'' \subseteq \hat{P}$ for every $\bar{G} \in \mathrm{GFan}_T(I)$.

Example 4.5. In $P = \mathbb{Q}[x,y,z,w]$, and let I be the ideal of P generated by $f_1 = xz - x + w^2$, $f_2 = w^2 + x - w$, and $f_3 = y^2 + z^2 + z$. Then the Gröbner fan GFan(I) has 12 marked reduced Gröbner bases. For $T = (y^2)$ we have $\mathcal{O} = \{1,y\}$ and $\hat{P} = \mathbb{Q}[x,z,w]$. The T-restricted Gröbner fan $\mathrm{GFan}_T(I)$ contains 3 elements, namely

$$\begin{aligned}
\overline{G}_1 &= \{(y^2, y^2 + z^2 + z), (w^2, w^2 + x - w), (xz, xz - 2x + w)\},\\
\overline{G}_2 &= \{(y^2, y^2 + z^2 + z), (x, x + w^2 - w), (zw^2, zw^2 - zw - 2w^2 + w)\},\\
\overline{G}_3 &= \{(y^2, y^2 + z^2 + z), (w, w + xz - 2x), (x^2z^2, x^2z^2 - 4x^2z + 4x^2 + xz - x)\}.
\end{aligned}$$

Observe that $G''_1 = \{w^2 + x - w, xz - 2x + w\}$, $G''_2 = \{x + w^2 - w, zw^2 - zw - 2w^2 + w\}$, and $G''_3 = \{w + xz - 2x, x^2z^2 - 4x^2z + 4x^2 + xz - x\}$. Clearly, $G''_i \subseteq \hat{P}$ for $i = 1,2,3$, and hence they form reduced Gröbner bases of the elimination ideal $I \cap \hat{P}$. In particular, we also have $\mathrm{GFan}(I \cap \hat{P}) = \{\overline{G''_1}, \overline{G''_2}, \overline{G''_3}\}$. In this case, $\mathcal{F} = \hat{P} \oplus y\hat{P}$, and we find the corresponding reduced σ′-Gröbner basis G^*_i of the submodule $I \cap \mathcal{F}$, for instance, with $i = 1$,

$$G'_1 = G^*_1 = \{w^2 + x - w, xz - 2x + w, y(w^2 + x - w), y(xz - 2x + w)\}.$$

Proposition 4.6. *Using the notation introduced in Proposition 4.3, if $G'' \subseteq \hat{P}$ for any $\bar{G} \in GFan_T(I)$, then the maps $\Gamma_T : GFan_T(I) \to GFan(I \cap \hat{P})$, $\bar{G} \mapsto \overline{G''}$ and $\Psi_T : GFan_T(I) \mapsto GFan(I \cap \mathcal{F})$, $\bar{G} \to \overline{G*}$ are injections.*

Proof. Clearly, the maps Γ_T and Ψ are well-defined by Proposition 4.3(b). Suppose that $\overline{G_1}, \overline{G_2} \in \mathrm{GFan}_T(I)$ satisfies $\Psi_T(\overline{G_1}) = \Psi_T(\overline{G_2}) = \overline{G*}$ (a similar proof for the case $\Gamma_T(\overline{G_1}) = \Gamma_T(\overline{G_2}) = \overline{G''}$). We write

$$\overline{G}_i = \{(t_1,\, t_1 - h_{i1}),\, \ldots,(t_s,\, t_s - h_{is}), (\mathrm{LT}_\sigma(g_{i\,s+1}),\, g_{i\,s+1}), \ldots, (\mathrm{LT}_\sigma(g_{i\,r_i}),\, g_{i\,r_i})\}$$

for $i = 1,2$. For $u = 1 \in \mathcal{O}_T$, we have

$$\{g_{1\,s+1},\ldots,g_{1\,r1}\} = \{g_{2\,s+1},\ldots,g_{2\,r2}\}$$

in G^*, and so $r_1 = r_2$. If $h_1k \neq h_2k$ then

$$f = (t_k - h_{1k}) - (t_k - h_{2k}) = h_{2k} - h_{1k} \in I \setminus \{0\}$$

and no term in the support of f is divisible by any leading term in $\mathrm{LT}_\sigma\{G\}$. Hence $\mathrm{LT}_\sigma(f) \notin \mathrm{LT}_\sigma(G)$, a contradiction to the fact that G is the reduced σ-Gröbner basis of I. Therefore, $t_k - h_{1k} = t_k - h_{2k}$ for $k = 1,\ldots,s$ and consequently $\overline{G_1} = \overline{G_2}$, as desired.

Example 4.7. Let us reconsider Example 4.5. For $T = (y^2)$, the T-restricted Gröbner fan $\mathrm{GFan}_T(I)$ contains 3 elements $\overline{G_1}$, $\overline{G_2}$, $\overline{G_3}$ which satisfy the assumption of Proposition 4.6. Hence the map $\Psi_T : \mathrm{GFan}_T(I) \to \mathrm{GFan}(I \cap \mathcal{F})$, $\bar{G_\iota} \mapsto \overline{G*_\iota}$ is injective. Especially, the map $\Gamma_T : \mathrm{GFan}_T(I) \to \mathrm{GFan}(I \cap \hat{P})$, $\bar{G_\iota} \mapsto \overline{G"_\iota}$ is a bijection.

Finally, we look at the case that the monomial ideal $\langle T \rangle := \langle t_1,\ldots,t_s\rangle$ is irreducible, i.e., if it cannot be written as proper intersection of two other monomial ideals. Note that under our assumption ti ∤ tj for all $i \neq j$, the tuple T is the minimal system of generators of $\langle T \rangle$.

Proposition 4.8. *The following conditions are equivalent:*

(a) The monomial ideal $\langle T\rangle$ is irreducible.

(b) There exist positive integers $e_1,\ldots,e_s$ and $1 \le i_1 < \cdots < i_s \le n$ such that $t_j = x_{ij}^{ej}$ for $j = 1,\ldots,s$.

(c) The tuple T satisfies $\sqrt{\langle T\rangle} = \langle Z\rangle$ and $\gcd(t_i,t_j) = 1$ for $i \neq j$.

If these conditions are satisfied, then $\mathcal{F}$ is a free $\hat{P}$-module of rank $\prod_{j=1}^{s} e_j$.

Proof. The equivalence of (a) and (b) follows from (Herzog & Hibi, 2011; Theorem 1.3.1) and the fact that T is the unique minimal system of generators of $\langle T\rangle$. Moreover, the implication (b) ⇒ (c) is clearly true. For the implication (c) ⇒ (b), we write $Z = \{x_{i1},\ldots,x_{ir}\}$ and see that $\sqrt{\langle T\rangle} = \langle Z\rangle$ yields $x_{ij}^{e'j} \in \langle T\rangle$ for $j = 1,\ldots,r$ and for some $e'_j \geq 1$. Then $x_{ij}^{e'j}$ is divisible by one element in T, say t_j, and hence $t_j = x_{ij}^{ej}$ for $1 \le e_j \le e'_j$. Thus $\{t_1,\ldots,t_r\} = \{x_{i1}^{e1},\ldots,x_{ir}^{er}\}$. If $s > r$ then the conditions $\gcd(t_j, t_{r+1}) = 1$ for all $j \in \{1,\ldots,r\}$ imply either $t_{r+1} = 1$ or $t_{r+1} \notin K[Z]$, and both cases yield a contradiction. Hence $s = r$ and $\{t_1,\ldots,t_r\} = \{t_1,\ldots,t_s\}$. Finally, the addition claim follows from the fact that $\#\,\mathcal{O}_T = \prod_{j=1}^{s} e_j$.

The condition (b) of the proposition also means that the monomial ideal $\langle T\rangle$ is generated by pure powers of some indeterminates.

Corollary 4.9. *Let $s \le n$, let $T = (t_1,\ldots,t_s)$ be a tuple of pure powers of indeterminates, let*

$$\overline{G} = \{(t_1,\, t_1 - h_1),\, \ldots,(t_s,\, t_s - h_s), (\mathrm{LT}_\sigma(g_{s+1}),\, g_{s+1}), \ldots, (\mathrm{LT}_\sigma(g_r),\, g_r)\}$$

be an element of $GFan_T(I)$, and let $G'' = \{g_{s+1},\ldots,g_r\}$.

(a) The set $\overline{H} = \{(t_1, t_1 - h_1), \ldots,(t_s, t_s - h_s)\}$ is an element of $GFan(\langle t_1 - h_1,\ldots,t_s - h_s\rangle)$. For $i = s+1,\ldots,r$, we have $g_i = NF_H(g_i)$.

(b) If $G'' \subseteq \hat{P}$ for any $\bar{G} \in GFan_T(I)$, then the maps $\Gamma_T : GFan_T(I) \to GFan(I \cap \hat{P})$, $\bar{G} \mapsto \overline{G"}$ and $\Psi_T : GFan_T(I) \to GFan(I \cap \mathcal{F})$, $\bar{G} \mapsto \overline{G*}$ are injections.

Proof. (a) We have $t_i = \mathrm{LT}_\sigma(t_i - h_i)$ for $i = 1,\ldots,s$. By Proposition 4.8, $\gcd(t_i, t_j) = 1$ for all $i \neq j$. Thus $\{t_1 - h_1,\ldots,t_s - h_s\}$ is a σ-Gröbner basis of the ideal $\langle t1 - h_1,\ldots,ts - hs\rangle$ by (Kreuzer & Robbiano, 2000, Proposition 2.5.8). Moreover, $\{t_1 - h_1,\ldots,t_s - h_s\}$ is fully reduced, as $\overline{\mathrm{G}} \in \mathrm{GFan}_T(I)$. Hence $\overline{H} \in \mathrm{GFan}(\langle t_1 - h_1,\ldots,t_s - h_s\rangle)$.

Moreover, it follows from that any term in the support of g_i is not divisible by t_i for $i = 1,\ldots,s$, and hence

$g_i = \mathrm{NF}_H(g_i)$ for $i = s+1,\ldots,r$.

(b) This follows from Propositions 4.3 and 4.6.

Example 4.10. Let us go back to Example 4.5. The Gröbner fan GFan(I) of $I = \langle xz - x + w^2, w^2 + x - w, y^2 + z^2 + z\rangle$ in $P = \mathbb{Q}[x,y,z,w]$ has 12 marked reduced Gröbner bases. For $T = (x)$, the T-restricted Gröbner fan GFan$_T(I)$ contains 3 elements, namely

$$\begin{aligned}
\overline{G}_1 = \{&(x, x + w^2 - w), (z^2, z^2 + y^2 + z), (zw^2, zw^2 - zw - 2w^2 + w),\\
&(y^2w^2, y^2w^2 - y^2w + zw + 6w^2 - 3w)\},\\
\overline{G}_2 = \{&(x, x + w^2 - w), (z^2, z^2 + z + y^2), (zw, zw + y^2w^2 - y^2w + 6w^2 - 3w)\\
&(y^2w^3, y^2w^3 - 2y^2w^2 + y^2w + 6w^3 - 7w^2 + 2w)\},\\
\overline{G}_3 = \{&(x, x + w^2 - w), (zw^2, zw^2 - zw - 2w^2 + w), (y^2, y^2 + z^2 + z)\}.
\end{aligned}$$

Also, one may check that GFan$_Z(I) = \emptyset$ for any pair Z in $\{x,y,z,w\}$.

• For $T_1 = (x,y^2)$, we find GFan$_{T1}(I) = \{\overline{G_3}\}$, in particular, $g_3 = zw^2 - zw - 2w^2 + w \in \hat{P} = \mathbb{Q}[z,w]$. Then $\{g_3\}$ is a reduced Gröbner basis of the elimination ideal $I \cap \hat{P}$. In this case, we have $\mathcal{O}_{T1} = \{1,y\}$ and $\mathcal{F} = \hat{P} \oplus y\hat{P}$. The corresponding reduced Gröbner basis of $I \cap \mathcal{F}$ is $G^*_3 = \{g_3, yg_3\}$, and the maps Γ_{T1} and Ψ_{T1} are injections.

• For $T_2 = (x,z^2)$, we have $\mathcal{O}_{T2} = \{1,z\}$, $\mathcal{F} = \hat{P} \oplus z\hat{P}$, and GFan$_{T2}(I) = \{\overline{G_1}, \overline{G_2}\}$, but $G''_1, G''_2 \not\subseteq \hat{P} = \mathbb{Q}[y,w]$. In this case, Γ_{T2} is not defined and $\{\overline{G^*_1}, \overline{G^*_2}\} \subseteq$ GFan($I \cap \mathcal{F}$), where

$$\begin{aligned}
\overline{G^*_1} = \{&(zw^2, zw^2 - zw - 2w^2 + w), (y^2w^2, y^2w^2 - y^2w + zw + 6w^2 - 3w)\}\\
\overline{G^*_2} = \{&(zw, zw + y^2w^2 - y^2w + 6w^2 - 3w),\\
&(y^2w^3, y^2w^3 - 2y^2w^2 + y^2w + 6w^3 - 7w^2 + 2w),\\
&(y^2zw^3, y^2zw^3 - 2y^2zw^2 + y^2zw + 6zw^3 - 7zw^2 + 2zw)\}.
\end{aligned}$$

The map Ψ_{T2} : GFan$_{T2}(I) \to$ GFan($I \cap \mathcal{F}$), $\overline{G_\iota} \mapsto \overline{G^*_\iota}$ is still injective.

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Ghi chú về các quạt Gröbner hạn chế

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TÓM TẮT

Trong bài báo này chúng tôi nghiên cứu về quạt Gröbner của một iđêan thực sự khác không trong một vành đa thức. Chúng tôi đưa ra mô tả cụ thể đối với quạt Gröbner hạn chế của iđêan ứng với một bộ các từ và chỉ ra cách tính toán nó. Hơn nữa, dưới một điều kiện xác định, chúng tôi chỉ ra một kết nối giữa các quạt Gröbner hạn chế của một iđêan và quạt Gröbner của mô đun loại bỏ tương ứng với iđêan.